Polynomials

By: Sagar Aggarwal

Contents

Introduction

- A polynomial of degree 1 is called a **linear polynomial.** For example, $2x - 3$, $\sqrt{3}x + 5$, $y + \sqrt{2}$, $x - \frac{2}{11}$, $3z + 4$, $\frac{2}{3}u + 1$, etc., are all linear polynomials.
- A polynomial of degree 2 is called a **quadratic polynomial.** The name ‘quadratic’ has been derived from the word ‘quadrate’, which means ‘square’. For example, $2x^2+3x-\frac{2}{5}$, y^2-2, $2-x^2+\sqrt{3}x$, $\frac{u}{3}-2u^2+5$, $\sqrt{5}v^2-\frac{2}{3}v$, $4z^2+\frac{1}{7}$ are quadratic polynomials. More generally, any quadratic polynomial in x is of the form ax^2+bx+c, where a, b, c are real numbers and $a \neq 0$.
- A polynomial of degree 3 is called a **cubic polynomial.** For example, $2-x^3$, x3, $\sqrt{2}x^3$, $3-x^2+x^3$, $3x^3-2x^2+x-1$ are cubic polynomials. The most general form of a cubic polynomial is ax^3+bx^2+cx+d, where a, b, c, d are real numbers and $a \neq 0$.

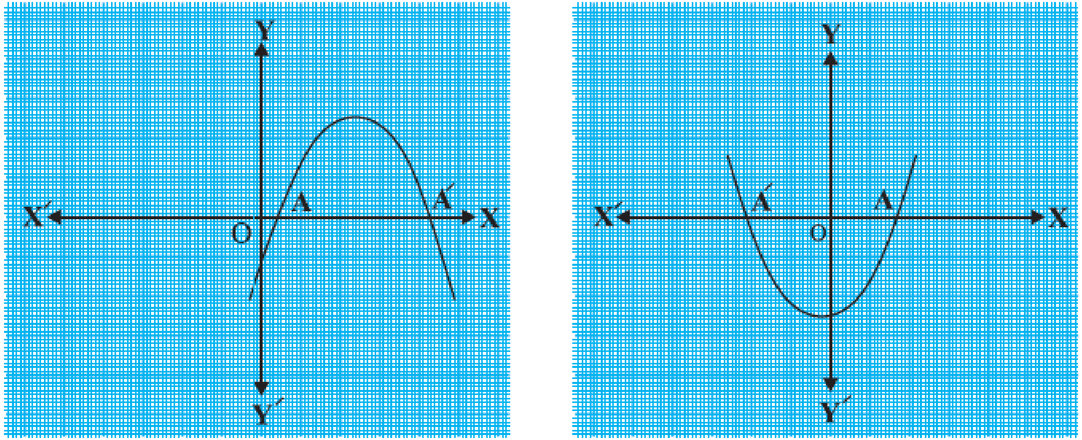
Introduction (Contd..)

- If p(x) is a polynomial in x, and if k is any real number, then the value obtained by replacing x by k in p(x), is called **the value of p(x) at x = k, and is denoted by p(k).** For example, let us consider a polynomial $p(x)=x^2-3x-4$. Then, the value of p(x) at x=2, is given by $p(2)=2^2-3(2)-4=4-6-4=-6$.
- A real number k is said to be a zero of a polynomial p(x), if p(k)=0. For example, let $p(x)=x^2-3x-4$. Now, $p(4)=4^2-3(4)-4=16-12-4=0$. So, 4 is the zero of polynomial $p(x)=x^2-3x-4$.
- If k is a zero of p(x)=ax+b, then p(k)=ak+b=0, i.e., $k=-\frac{b}{a}$. So the zero of the linear polynomial ax+b is $-\frac{b}{a}=\frac{-(\text{Constant term})}{\text{Coefficient of } x}$. For example, if k is a zero of p(x)=2x+3, then p(k)=0 gives us 2k+3=0, i.e., $k=-\frac{3}{2}$.

Geometrical Meaning of the Zeroes of a Polynomial

- The following three cases can happen about the shape of the graph $y=ax^2+bx+c$:

  Case (i): Here, the graph cuts x-axis at two distinct points A and A′**.** The x-coordinates of A and A′ are the **two zeroes** of the quadratic polynomial ax^2+bx+c in this case (see Fig. 1.1).

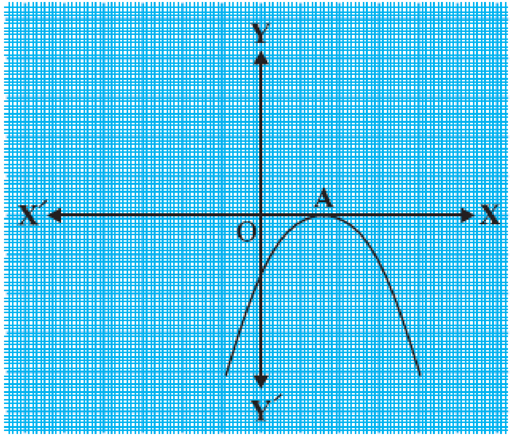


(i) (ii)

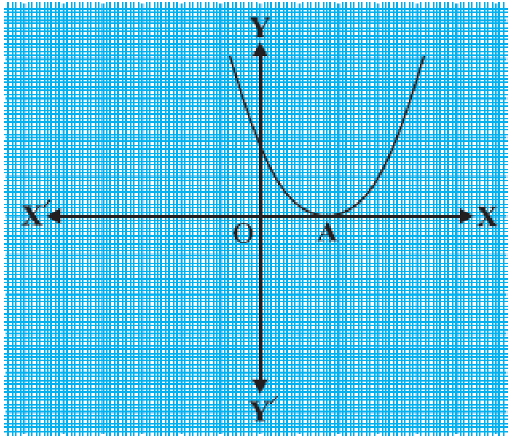
Fig. 1.1

Geometrical Meaning of the Zeroes of a Polynomial (Contd..)

Case (ii): Here, the graph cuts the x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A′ of Case (i) coincide here to become one point A (see Fig. 1.2). The x-coordinate of A is the **only zero** for the quadratic polynomial ax^2+bx+c in this case.



(i)



(ii)

Fig. 1.2

Geometrical Meaning of the Zeroes of a Polynomial (Contd..)

Case (iii): Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point (see Fig. 1.3). So, the quadratic polynomial ax^2+bx+c has **no zero** in this case.

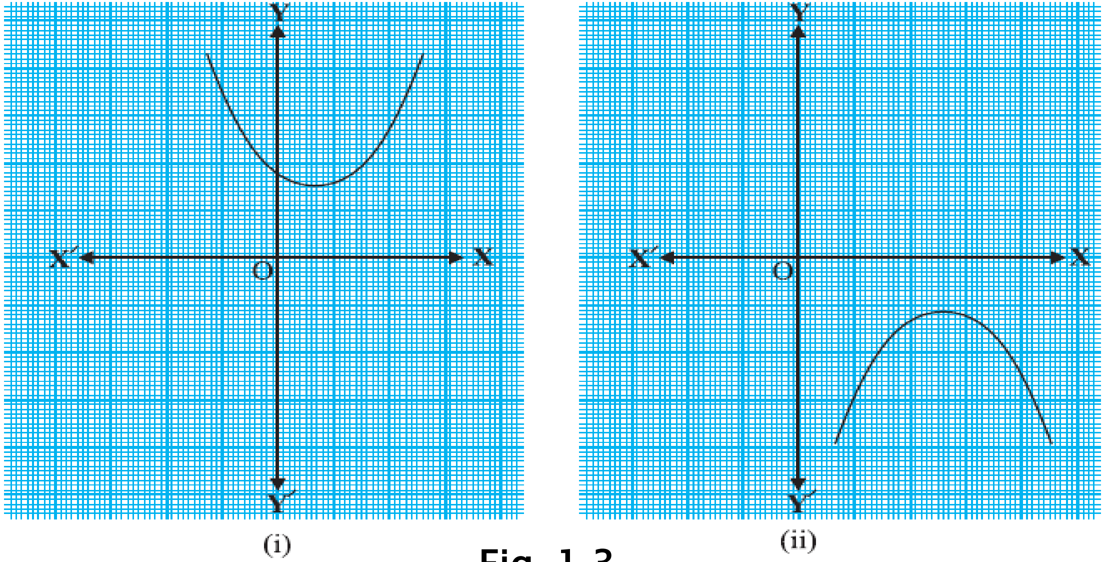


Fig. 1.3

Geometrical Meaning of the Zeroes of a Polynomial (Contd..)

- We can try to draw the graph of cubic equations. We will find that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.
- **Remark**: In general, given a polynomial p(x) of degree n, the graph of y = p(x) intersects the x-axis at atmost n points. Therefore, a polynomial p(x) of degree n has at most n zeroes.

Relationship between Zeroes and Coefficients of a Polynomial

- In general, if α* and β* are the zeroes of the quadratic polynomial $p(x)=ax^2+bx+c$, $a\neq0$, then you know that $x-\alpha$ and $x-\beta$ are the factors of $p(x)$. Therefore,

  $ax^2+bx+c=k(x-\alpha)(x-\beta)$, where k is a constant

  $=k[x^2-(\alpha+\beta)x+\alpha\beta]$

  $=kx^2-k(\alpha+\beta)x+k\alpha\beta$

  Comparing the coefficients of x^2, x and constant terms on both the sides, we get

  $a=k$, $b=-k(\alpha+\beta)$ and $c=k\alpha\beta$.

  This gives sum of zeroes $=\alpha+\beta=-\dfrac{b}{a}=\dfrac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$

  product of zeroes $=\alpha\beta=\dfrac{c}{a}=\dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

*α, β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

Relationship between Zeroes and Coefficients of a Polynomial (Contd..)

- **Example 1**: Find the zeroes of the quadratic polynomial $x^2+7x+10$, and verify the relationship between the zeroes and the coefficients.

  Solution: We have $x^2+7x+10=(x+2)(x+5)$

  So, the value of $x^2+7x+10$ is zero when $x+2=0$ or $x+5=0$, i.e., when $x=-2$ or $x=-5$. Therefore, the zeroes of $x^2+7x+10$ are -2 and -5. Now,

  Sum of zeroes $=-2+(-5)=-7=\dfrac{-(7)}{1}=\dfrac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$

  Product of zeroes $=(-2)\text{x}(-5)=10=\dfrac{10}{1}=\dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Relationship between Zeroes and Coefficients of a Polynomial (Contd..)

- In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial ax^3+bx^2+cx+d, then

  Sum of zeroes $=\alpha+\beta+\gamma=\frac{-b}{a}=\frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}$

  Sum of the products of zeroes taken two at a time $=\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

  Product of zeroes $=\alpha\beta\gamma=\frac{-d}{a}=\frac{-(\text{Constant term})}{\text{Coefficient of } x^3}$

Relationship between Zeroes and Coefficients of a Polynomial (Contd..)

- **Example 2***: Verify that 3, -1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x)=3x^3-5x^2-11x-3$,

  and then verify the relationship between the zeroes and the coefficients.

  Solution: Comparing the given polynomial with ax^3+bx^2+cx+d, we get

  $a=3$, $b=-5$, $c=-11$, $d=-3$. Further

  $p(3)=3\times3^3-(5\times3^2)-(11\times3)-3=81-45-33-3=0$,

  $p(-1)=3\times(-1)^3-5\times(-1)^2-11\times(-1)-3=-3-5+11-3=0$,

$$p\left(-\frac{1}{3}\right)=3\times\left(-\frac{1}{3}\right)^3-5\times\left(-\frac{1}{3}\right)^2-11\times\left(-\frac{1}{3}\right)-3$$

$$=-\frac{1}{9}-\frac{5}{9}+\frac{11}{3}-3=-\frac{2}{3}+\frac{2}{3}=0$$

* Not from the examination point of view.

Relationship between Zeroes and Coefficients of a Polynomial (Contd..)

Therefore, 3, -1 and $-\frac{1}{3}$ are the zeroes of $3x^3-5x^2-11x-3$.

So, we take $\alpha=3$, $\beta=-1$ and $\gamma=-\frac{1}{3}$

Now,

$$\alpha+\beta+\gamma=3+(-1)+\left(-\frac{1}{3}\right)=3-1-\frac{1}{3}=\frac{9-3-1}{3}=\frac{5}{3}=\frac{-(-5)}{3}=\frac{-b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=3\times(-1)+(-1)\times\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\times 3=-3+\frac{1}{3}-1=\frac{-11}{3}=\frac{c}{a}$$

$$\alpha\beta\gamma=3\times(-1)\times\left(-\frac{1}{3}\right)=1=\frac{1}{1}=\frac{-(-3)}{3}=\frac{-d}{a}$$

Division Algorithm for Polynomials

- We know that, Dividend=Divisor×Quotient+Remainder

  Division Algorithm for polynomials says that

  If p(x) and g(x) are any two polynomials with g(x) ≠ 0, then we can find polynomials q(x) and r(x) such that

  $$p(x)=g(x)\times q(x)+r(x),$$

  where r(x)=0 or degree of r(x)<degree of g(x).

- **Example 3**: Divide $3x^2-x^3-3x+5$ by $x-1-x^2$, and verify the division algorithm.

  Solution: Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

  So, dividend$=-x^3+3x^2-3x+5$ and

  divisor$=-x^2+x-1$.

$$\begin{array}{r|l} & x-2 \\ \hline -x^2+x-1 & -x^3+3x^2-3x+5 \\ & -x^3+\ x^2-\ x \\ & +\quad -\quad + \\ \hline & 2x^2-2x+5 \\ & 2x^2-2x+2 \\ & -\quad +\quad - \\ \hline & 3 \end{array}$$

Division Algorithm for Polynomials (Contd..)

Division process is shown on the right side.

We stop here since degree (3)=0<2=degree ($-x^2+x-1$).

So, quotient=x–2, remainder=3.

Now,

Divisor×Quotient+Remainder

$=(-x^2+x-1)(x-2)+3$

$=-x^3+x^2-x+2x^2-2x+2+3$

$=-x^3+3x^2-3x+5$

=Dividend

In this way, the division algorithm is verified.

Division Algorithm for Polynomials (Contd..)

- **Example 4**: Find all the zeroes of 2x4 – 3x3 – 3x2 + 6x – 2, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

  Solution: Since two zeroes are $\sqrt{2}$ and $-\sqrt{2}$, $(x-\sqrt{2})(x+\sqrt{2})=x^2-2$ is a factor of the given polynomial. Now, we divide the given polynomial by x^2-2.

$$2x^2-3x+1$$

$$x^2-2 \,\big)\, 2x^4-3x^3-3x^2+6x-2$$

$$2x^4 \qquad -4x^2$$

$$- \qquad\qquad +$$

$$-3x^3+x^2+6x-2$$

$$-3x^3 \qquad +6x$$

$$+ \qquad\qquad -$$

$$x^2 \qquad -2$$

$$x^2 \qquad -2$$

$$- \qquad\qquad +$$

$$0$$

First term of quotient is $\dfrac{2x^4}{x^2}=2x^2$

Second term of quotient is $\dfrac{-3x^3}{x^2}=-3x$

Third term of quotient is $\dfrac{x^2}{x^2}=1$

Division Algorithm for Polynomials (Contd..)

So, $2x^4-3x^3-3x^2+6x-2=(x^2-2)(2x^2-3x+1)$.

Now, by splitting $-3x$, we factorise $2x^2-3x+1$ as $(2x-1)(x-1)$. So, its zeroes are given by $x=\frac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2}$, $-\sqrt{2}$, $\frac{1}{2}$ and 1.

Summary

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in x with real coefficients is of the form ax^2+bx+c, where a, b, c are real numbers with $a\neq0$.
3. The zeroes of a polynomial p(x) are precisely the x-coordinates of the points, where the graph of $y=p(x)$ intersects the x-axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at most 3 zeroes.
5. If α and β are the zeroes of the quadratic polynomial ax^2+bx+c, then

$$\alpha+\beta=\frac{-b}{a}, \quad \alpha\beta=\frac{c}{a}$$

Summary (Contd..)

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3+bx^2+cx+d=0$, then

$$\alpha+\beta+\gamma=\frac{-b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$

and $$\alpha\beta\gamma=\frac{-d}{a}$$

7. The division algorithm states that given any polynomial p(x) and any non-zero polynomial g(x), there are polynomials q(x) and r(x) such that

$$p(x)=g(x)q(x)+r(x),$$

where $r(x)=0$ or degree $r(x)<$ degree $g(x)$.

THANK YOU